Practice Problems on Transformations of Random Variables

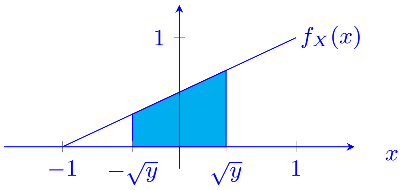
Math 262

1. Let X have pdf given by $f_X(x) = \frac{x+1}{2}$ for $-1 \le x \le 1$. Find the density of $Y = X^2$.

Note that $0 \le Y \le 1$. Fix $y \in [0,1]$. Then:

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y})$$

This probability is equal to the shaded area below:



The shaded region is a trapezoid with area $\sqrt{y}$, so $F_Y(y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \sqrt{y}$. Differentiating, we find $f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{1}{2\sqrt{y}}$ for $0 \le y \le 1$.

2. Let Y have pdf given by $f_Y(y) = 2(1-y)$ for $0 \le y \le 1$.

(a) Find the density of $U_1 = 2Y - 1$.

$U_1 = g_1(Y)$, where $g_1(y) = 2y - 1$. Since g_1 is monotonic, we can apply the Transformation Theorem. The inverse of g_1 is $h_1(u_1) = \frac{u_1+1}{2}$ for $-1 \le u_1 \le 1$. The density of U_1 is then:

$$f_{U_1}(u_1) = f_Y(h_1(u_1))\left|h_1'(u_1)\right| = 2\left(1 - \frac{u_1+1}{2}\right)\left|\frac{1}{2}\right| = \frac{1-u_1}{2} \qquad \text{for } -1 \le u_1 \le 1.$$

(b) Find the density of $U_2 = 1 - 2Y$.

$U_2 = g_2(Y)$, where $g_2(y) = 1 - 2y$. Since g_2 is monotonic, we can apply the Transformation Theorem. The inverse of g_2 is $h_2(u_2) = \frac{1-u_2}{2}$, for $-1 \le u_2 \le 1$. The density of U_2 is then:

$$f_{U_2}(u_2) = f_Y(h_2(u_2))\left|h_2'(u_2)\right| = 2\left(1 - \frac{1-u_2}{2}\right)\left|\frac{1}{2}\right| = \frac{1+u_2}{2} \qquad \text{for } -1 \le u_2 \le 1.$$

(c) Find the density of $U_3 = Y^2$.

$U_3 = g_3(Y)$, where $g_3(y) = y^2$, which *is* monotonic on the interval $0 \le y \le 1$, so we can apply the Transformation Theorem. The inverse of g_3 is $h_3(u_3) = \sqrt{u_3}$, for $0 \le u_3 \le 1$. The density of U_3 is then:

$$f_{U_3}(u_3) = f_Y(h_3(u_3))\left|h_3'(u_3)\right| = 2\left(1 - \sqrt{u_3}\right)\left|\frac{1}{2\sqrt{u_3}}\right| = \frac{1}{\sqrt{u_3}} - 1 \qquad \text{for } 0 \le u_3 \le 1.$$

3. Let $X \sim \text{Unif}[0,1]$. Find the density of $U = \sqrt{X}$.

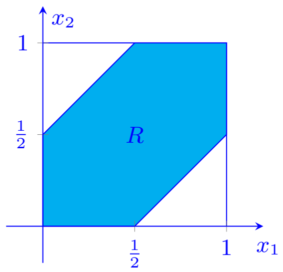
First, $f_X(x) = 1$ for $0 \le x \le 1$. Since $U = g(X)$, where $g(x) = \sqrt{x}$, which is monotonic, we can apply the Transformation Theorem. The inverse of g is $h(u) = u^2$ for $0 \le u \le 1$. The density of U is then:

$$f_U(u) = f_X(h(u))\left|h'(u)\right| = 1\,|2u| = 2u \qquad \text{for } 0 \le u \le 1.$$

4. Two sentries are sent to patrol a road that is 1 mile long. The sentries are sent to points chosen independently and uniformly along the road. Find the probability that the sentries will be less than $\frac{1}{2}$ mile apart when they reach their assigned posts.

Let X_1 and X_2 be the posts of the sentries along the road; X_1 and X_2 are iid Unif$[0, 1]$. Thus, their joint density is $f(x_1, x_2) = 1$ for $0 \le x_1 \le 1$ and $0 \le x_2 \le 1$.

Let $Y = X_1 - X_2$. We want $P\left(-\frac{1}{2} < Y < \frac{1}{2}\right)$. Note that we don't need the density of Y to answer the question: since the joint density of X_1 and X_2 is constant on the unit square, the probability $P\left(-\frac{1}{2} < Y < \frac{1}{2}\right)$ is equal to the area of the shaded region R in the following figure.



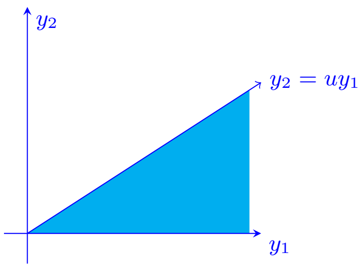
Thus, $P\left(-\frac{1}{2} < Y < \frac{1}{2}\right) = \frac{3}{4}$.

5. The joint distribution for the lifetimes of two different types of components operating in a system is given by

$$f(y_1, y_2) = \begin{cases} \frac{1}{8} y_1 e^{-(y_1+y_2)/2} & \text{if } y_1 > 0, y_2 > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find the density function for the ratio $U = \frac{Y_2}{Y_1}$.

Using the (bivariate) distribution function method, first note that U can be any positive number. Fix $u > 0$, and note that the set of where $U = \frac{Y_2}{Y_1} = u$ in the $y_1 y_2$-plane is the line $y_2 = u y_1$.



The region where $U = \frac{Y_2}{Y_1} \le u$ is the region in the first quadrant where $y_2 \le u y_1$, which is the shaded region in the figure above.

Then, $P(U \le u) = P\left(\frac{Y_2}{Y_1} \le u\right) = \int_0^\infty \int_0^{uy_1} \frac{1}{8} y_1 e^{-(y_1+y_2)/2} dy_2 dy_1 = \frac{u^2 + 2u}{(1+u)^2}$.

Thus, the density of U is $f_U(u) = \frac{d}{du}\left(\frac{u^2 + 2u}{(1+u)^2}\right) = \frac{2}{(1+u)^3}$, for $u > 0$.

6. Suppose X and Y are independent exponential rvs with parameter λ. Find the joint density of $V = \frac{X}{Y}$ and $W = X + Y$. Use the joint density to find the marginal distributions.

We will use the bivariate tranformation theorem. Note that the joint density of X and Y is given by $f(x, y) = \lambda^2 e^{-\lambda(x+y)}$ for $x > 0$ and $y > 0$.

We must solve for X and Y in terms of V and W. Since $V = \frac{X}{Y}$, it follows that $X = VY$, and then $W = X+Y = VY+Y$, which we solve for Y to obtain $Y = \frac{W}{V+1}$. Similarly, we find $X = \frac{VW}{V+1}$. Thus, we have $X = \phi(V, W)$ where $\phi(v, w) = \frac{vw}{v+1}$, and $Y = \psi(V, W)$ where $\psi(v, w) = \frac{w}{v+1}$.

The Jacobian determinant is then:

$$|M| = \begin{vmatrix} \frac{\partial \phi}{\partial v} & \frac{\partial \phi}{\partial w} \\ \frac{\partial \psi}{\partial v} & \frac{\partial \psi}{\partial w} \end{vmatrix} = \begin{vmatrix} \frac{w}{(v+1)^2} & \frac{v}{v+1} \\ \frac{-w}{(v+1)^2} & \frac{1}{v+1} \end{vmatrix} = \frac{w}{(v+1)^3} - \frac{-vw}{(v+1)^3} = \frac{w}{(v+1)^2}.$$

Therefore, the joint density of V and W is given by

$$g(v, w) = f(\phi(v, w), \psi(v, w))|M| = \lambda^2 e^{-\lambda\left(\frac{vw}{v+1} + \frac{w}{v+1}\right)} \left| \frac{w}{(v+1)^2} \right| = \frac{\lambda^2 w}{(v+1)^2} e^{-\lambda w}$$

for $v > 0$ and $w > 0$. Integrate to find the marginal densities:

$$g_V(v) = \int_0^\infty g(v, w)\, dw = \frac{1}{(1+v)^2}$$

and

$$g_W(w) = \int_0^\infty g(v, w)\, dv = \lambda^2 w e^{-\lambda w}.$$

7. Let X and Y have joint density $f(x, y)$. Let (R, Θ) be the polar coordinates of (X, Y).

(a) Give a general expression for the joint density of R and Θ.

Note that $R = \sqrt{X^2 + Y^2}$, $\Theta = \arctan\left(\frac{Y}{X}\right)$, $X = R\cos\Theta$, and $Y = R\sin\Theta$.

The Jacobian determinant is then:

$$|M| = \begin{vmatrix} \frac{\partial}{\partial r} r\cos\theta & \frac{\partial}{\partial \theta} r\cos\theta \\ \frac{\partial}{\partial r} r\sin\theta & \frac{\partial}{\partial \theta} r\sin\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

The joint density of R and Θ is given by:

$$g(r, \theta) = f(r\cos\theta, r\sin\theta)|M| = f(r\cos\theta, r\sin\theta)r.$$

(b) Suppose X and Y are independent with $f(x) = 2x$ for $0 < x < 1$ and $f(y) = 2y$ for $0 < y < 1$. Use your result to find the probability that (X, Y) lies inside the circle of radius 1 centered at the origin.

The joint density of X and Y is given by $f(x, y) = 4xy$ for $0 < x < 1$ and $0 < y < 1$.

By the previous result, the joint density of R and Θ is given by

$$g(r, \theta) = f(r\cos\theta, r\sin\theta)r = 4(r\cos\theta)(r\sin\theta)r = 4r^3\cos\theta\sin\theta.$$

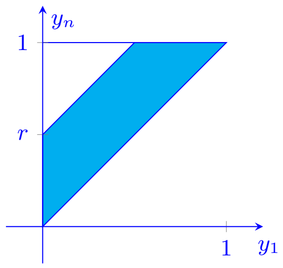
The point (X, Y) lies within the unit circle if and only if $R < 1$. Since both X and Y are positive, $0 < \Theta < \frac{\pi}{2}$, so the probability that $R < 1$ is given by

$$P(R < 1) = \int_0^{\pi/2} \int_0^1 4r^3 \cos\theta \sin\theta \, dr d\theta = \frac{1}{2}.$$

8. Let $X_1, X_2, \ldots, X_n$ denote a random sample from the uniform distribution on $[0, 1]$. Let Y_1 and Y_n be the smallest and largest, respectively, among the X_i. Find the pdf for the range $R = Y_n - Y_1$.

Hint: The joint pdf for Y_1 and Y_n is $g(y_1, y_n) = n(n-1)(y_n - y_1)^{n-2}$ for $0 \le y_1 \le y_n \le 1$. (See exercise 141 in Chapter 4 of Carlton and Devore.)

Since $0 \le R \le 1$, fix $r \in [0, 1]$. Then $R = Y_n - Y_1 = r$ along the line $y_n = y_1 + r$ in the $y_1 y_n$-plane. Furthermore, $R \le r$ in the region below this line, which is the shaded region in the following diagram.



Thus, the cdf of R is:

$$F_R(r) = P(R \le r) = 1 - \int_r^1 \int_0^{y_n - r} n(n-1)(y_n - y_1)^{n-2}\, dy_1 dy_n = (1-n)r^n + nr^{n-1}.$$

Differentiate to find the pdf of R:

$$f_R(r) = \frac{d}{dr} F_R(r) = \frac{d}{dr}\left[(1-n)r^n + nr^{n-1}\right] = n(n-1)\left(r^{n-2} - r^{n-1}\right) \qquad \text{for } 0 \le r \le 1.$$

Example 23-1

Suppose X_1 and X_2 are independent exponential random variables with parameter $\lambda = 1$ so that

$$f_{X_1}(x_1) = e^{-x_1} \qquad 0 < x_1 < \infty$$
$$f_{X_2}(x_2) = e^{-x_2} \qquad 0 < x_2 < \infty$$

The joint pdf is given by

$$f(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = e^{-x_1 - x_2} \qquad 0 < x_1 < \infty, 0 < x_2 < \infty$$

Consider the transformation: $Y_1 = X_1 - X_2, Y_2 = X_1 + X_2$. We wish to find the joint distribution of Y_1 and Y_2.

We have

$$x_1 = \frac{y_1 + y_2}{2}, x_2 = \frac{y_2 - y_1}{2}$$

OR

$$v_1(y_1, y_2) = \frac{y_1 + y_2}{2}, v_2(y_1, y_2) = \frac{y_2 - y_1}{2}$$

The Jacobian, J is

$$\begin{vmatrix} \frac{\partial\left(\frac{y_1+y_2}{2}\right)}{\partial y_1} & \frac{\partial\left(\frac{y_1+y_2}{2}\right)}{\partial y_2} \\ \frac{\partial\left(\frac{y_2-y_1}{2}\right)}{\partial y_1} & \frac{\partial\left(\frac{y_2-y_1}{2}\right)}{\partial y_2} \end{vmatrix}$$
$$= \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

So,

$$\begin{aligned} g(y_1, y_2) &= e^{-v_1(y_1,y_2) - v_2(y_1,y_2)} \left|\frac{1}{2}\right| \\ &= e^{-\left[\frac{y_1+y_2}{2}\right] - \left[\frac{y_2-y_1}{2}\right]} \left|\frac{1}{2}\right| \\ &= \frac{e^{-y_2}}{2} \end{aligned}$$

Now, we determine the support of (Y_1, Y_2). Since $0 < x_1 < \infty, 0 < x_2 < \infty$, we have $0 < \frac{y_1+y_2}{2} < \infty, 0 < \frac{y_2-y_1}{2} < \infty$ or $0 < y_1 + y_2 < \infty, 0 < y_2 - y_1 < \infty$. This may be rewritten as $-y_2 < y_1 < y_2, 0 < y_2 < \infty$.

Using the joint pdf, we may find the marginal pdf of Y_2 as

$$\begin{aligned} g(y_2) &= \int_{-\infty}^{\infty} g(y_1, y_2) dy_1 \\ &= \int_{-y_2}^{y_2} \frac{1}{2} e^{-y_2} dy_1 \\ &= \frac{1}{2}\left[e^{-y_2} y_1 \Big|_{y_1=-y_2}^{y_1=y_2}\right] \\ &= \frac{1}{2} e^{-y_2}(y_2 + y_2) \\ &= y_2 e^{-y_2}, \qquad 0 < y_2 < \infty \end{aligned}$$

Similarly, we may find the marginal pdf of Y_1 as

$$g(y_1) = \begin{cases} \int_{-y_1}^{\infty} \frac{1}{2} e^{-y_2} dy_2 = \frac{1}{2} e^{y_1} & -\infty < y_1 < 0 \\ \int_{y_1}^{\infty} \frac{1}{2} e^{-y_2} dy_2 = \frac{1}{2} e^{-y_1} & 0 < y_1 < \infty \end{cases}$$

Equivalently,

$$g(y_1) = \frac{1}{2} e^{-|y_1|} 0 < y_1 < \infty$$

This pdf is known as the **double exponential** or **Laplace** pdf.

WEEK 5

5. Transformations of Random Variables

We know that a random variable is a function from the sample space to the real number. That is, if X is a random variable it is a function from Ω to $\mathbb{R}$ ($X:\Omega \to \mathbb{R}$). The range of a random variable is a subset of the real numbers. As we know, if the range of the random variable D_X is a countable subset of the real number then it is called a discrete random variable and it is continuous otherwise. Now, consider a function g from $\mathbb{R}$ to $\mathbb{R}$ ($g:\mathbb{R} \to \mathbb{R}$). The composite function $g \circ X$ is also a function from sample space to real numbers ($g \circ X:\Omega \to \mathbb{R}$) and therefore $g \circ X$ is also a random variable (see Figure 1).

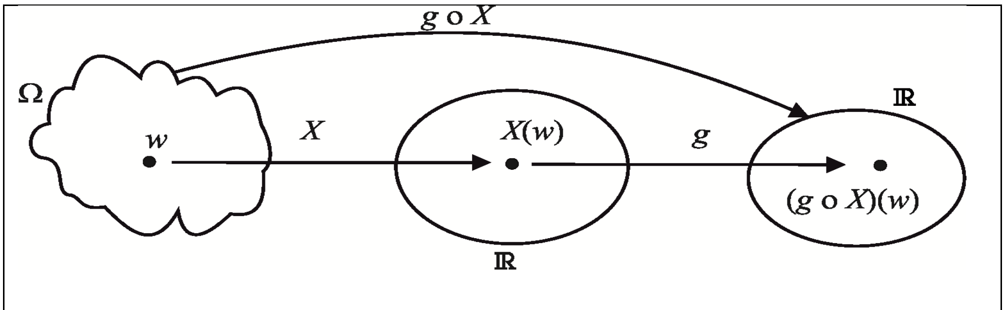


Figure 1. Transformation of random variable

The composite function $g \circ X$ is sometimes denoted as $g(X)$ and it is defined as for any $w \in \Omega$, $(g \circ X)(w) = g(X(w))$. Moreover, the range of $Y = g(X)$ is also a subset of $\mathbb{R}$, $D_Y \subset \mathbb{R}$.

If the random variable X is continuous, the transformed random variable $g(X)$ (say Y) may be either continous or discrete. Similarly, when X is discrete $g(X)$ may be discrete or continuous. In our study, if X is continuous $g(X)$ will be continuous and if X is discrete $g(X)$ will be discrete otherwise citeted.

In this part of the class, our goal is to find the distribution of the transformed random variable. Later, we are going to investigate the multivariate version of the transformations.

A) **<u>Discrete Case</u>**:

In discrete case, the easiest way to find the distribution of the transformed random variable is to calculate the probabilities directly.

Example: **a)** Let X be a random variable with the following probability function:

$$f(x)=\begin{cases} c & , \quad x=-2,-1,0,1,2 \\ 0 & , \quad elsewhere. \end{cases}$$

Note that from the range of X is $D_X=\{-2,-1,0,1,2\}$. Note that since

$$1=\sum_{x\in D_X} f(x)=\sum_{x=-2}^{2} c=5c$$

we have $c=1/5$ because $5c=1 \Rightarrow c=1/5$. That is the probability distribution of X is

$$f(x)=\begin{cases} 1/5 & , \quad x=-2,-1,0,1,2 \\ 0 & , \quad elsewhere. \end{cases}$$

Now, we want to find the probability distribution of the random variable $Y=X^2$. Note that the range of Y is $D_Y=\{0,1,4\}$ and the corresponding probabilities are calculated as

$$P(Y=0)=P(X=0)=\frac{1}{5}$$

$$P(Y=1)=P(X^2=1)=P(X=-1 \text{ or } X=1)=P(X=-1)+P(X=1)=\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$$

$$P(Y=4)=P(X^2=4)=P(X=-2 \text{ or } X=2)=P(X=-2)+P(X=2)=\frac{1}{5}+\frac{1}{5}=\frac{2}{5}$$

and therefore the probabilithy function of X and Y are given below:

$X=x$	-2	-1	0	1	2
$P(X=x)$	$1/5$	$1/5$	$1/5$	$1/5$	$1/5$

$Y=y$	0	1	2
$P(Y=y)$	$1/5$	$2/5$	$2/5$

b) Now let X be a random variable with the following probability function.

$X=x$	-3	0	1	2	3
$P(X=x)$	$3/8$	$1/8$	$1/8$	$1/8$	$2/8$

Suppose we want to find the distribution of $Y=X^2$ as before. Note that the range of the random variables are $D_X=\{-3,0,1,2,3\}$ and $D_Y=\{0,1,4,9\}$. Note that the probabilities of Y are calculated as

$$P(Y=0)=P(X^2=0)=P(X=0)=\frac{1}{8}, \quad P(Y=1)=P(X^2=1)=P(X=1)=\frac{1}{8}$$

$$P(Y=4)=P(X^2=4)=P(X=2)=\frac{1}{8}, \quad P(Y=9)=P(X^2=9)=P(X=-3)+P(X=3)=\frac{3}{8}+\frac{2}{8}=\frac{5}{8}$$

and the probability distribution can be written as

$Y=y$	0	1	4	9
$P(Y=y)$	$1/8$	$1/8$	$1/8$	$5/8.$

c) Now consider the random variable given in part (b) and find the probability distribution of $Y=2X+1$. Note that the range of the random variables are

$D_X=\{-3,\ 0\ ,1\ ,2\ ,3\}$ and $D_Y=\{-5\ ,1\ ,3\ ,5,\ 7\}$.

Similarly, the probabilities can be calculated as

$$P(Y=-5)=P(2X+1=-5)=P(X=-3)=\frac{3}{8},\qquad P(Y=1)=P(2X+1=1)=P(X=0)=\frac{1}{8}$$

$$P(Y=3)=P(2X+1=3)=P(X=1)=\frac{1}{8},\qquad P(Y=5)=P(2X+1=5)=P(X=2)=\frac{1}{8}$$

$$P(Y=7)=P(2X+1=7)=P(X=3)=\frac{2}{8}$$

and therefore the probability distribution can be written as

$Y=y$	-5	1	3	5	7
$P(Y=y)$	$3/8$	$1/8$	$1/8$	$1/8$	$2/8.$

B) <u>Continuous Case</u>:

Remember that the probability density function of a continuous ramdom variable is the derivative of the cummulative distribution function. Thus, if we can find the distribution of the transformed random variable, we can derivate it to find the probability density function of the transformen random variable.

Let X be a continuous random variable with probability density function $f(x)$, cummulative distribution function $F(x)$ with the range D_X. Consider a transformed random variable $Y=g(X)$. At this moment we assume that the function g is differentiable. The cummulative distribution function of Y can be calculated for all $y\in D_Y$ as

$$F_Y(y)=P(Y\le y)=P(g(X)\le y)=P(X\le g^{-1}(y))=F_X(g^{-1}(y))\ .$$

Thus, the probability density function of the transformed random variable Y is the derivative of $F_Y(y)$ which is

$$f_Y(y)=\frac{dF_Y(y)}{dy}=\frac{d}{dy}\Big[F_X(g^{-1}(y))\Big]=f_X(g^{-1}(y))\frac{d}{dy}\Big[g^{-1}(y)\Big].$$

Note that the derivative of $g^{-1}(y)$ may be negative and the probability can not be a negative number. For example if g is a decreasing function the derivative is negative. Therefore, we take the absolute value of the derivative (this derivative is known as the jacobien) and thus the probability density function of the transformed random variable can be written as

$$\boxed{f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}\left[g^{-1}(y)\right] \right| .} \qquad (1)$$

<u>Example 1</u>: Let X be a random variable with the following probability density function

$$f(x) = \begin{cases} c\,x & , \quad 0 < x < 1 \\ 0 & , \quad elsewhere. \end{cases}$$

a) The constant c can be determined from

$$1 = \int_{x=0}^{1} f(x)\,dx = c \int_{x=0}^{1} x\,dx = c\left.\frac{x^2}{2}\right|_{x=0}^{1} = \frac{c}{2} \Rightarrow \boxed{c = 2.}$$

b) Let us find the probability density function of $Y = 2X + 1$. Obviously, since $D_X = (0,1)$ the range of Y is $D_Y = (1,3)$. Thus, $F_Y(y) = 0$ for $y \le 1$ and $F_Y(y) = 1$ for $y \ge 3$. Now, for $1 < y < 3$ the cummulative distribution function

$$F_Y(y) = P(Y \le y) = P(2X + 1 \le y) = P(X \le (y-1)/2) = \int_{x=0}^{(y-1)/2} 2x\,dx = x^2\Big|_{x=0}^{(y-1)/2} = \left(\frac{y-1}{2}\right)^2 = \frac{(y-1)^2}{4}.$$

That is, the cummulative distribution function and the probability density function of $Y = 2X + 1$ are

$$F_Y(y) = \begin{cases} 0 & , \quad y \le 1 \\ (y-1)^2/4 & , \quad 0 < y < 3 \\ 1 & , \quad y \ge 3 \end{cases} , \qquad f_Y(y) = \frac{d}{dy}F_Y(y) = \begin{cases} \dfrac{(y-1)}{2} & , \quad 1 < y < 3 \\ 0 & , \quad elsewhere. \end{cases}$$

This is a probability density function because

$$\int_{y=1}^{3} f_Y(y)\,dy = \frac{1}{2}\int_{y=1}^{3} (y-1)\;d\,y = \left.\left(\frac{y^2}{4} - \frac{y}{2}\right)\right|_{y=1}^{3} = \left(\frac{9}{4} - \frac{3}{2}\right) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{3}{4} + \frac{1}{4} = 1.$$

The same probability density function can be found by using the equation (1). Note that $y = g(x) = 2x + 1 \Rightarrow x = (y-1)/2$. That is, $g^{-1}(y) = (y-1)/2$ and the derivative of this inverse function is

$$\frac{d}{dy}\left[g^{-1}(y)\right] = \frac{d}{dy}\left(\frac{y-1}{2}\right) = \frac{1}{2}$$

and using the equation (1) we write the probability density function of Y for $1 < y < 3$ as

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy}\left[g^{-1}(y)\right] \right| = 2\left(\frac{y-1}{2}\right) \left| \frac{1}{2} \right| = \frac{y-1}{2}$$

which is the same as above.

c) Now, let us try to find the probability density function of $Y=-2X+1$. Note that the range of Y is $D_Y=(-1\,,\,1)$. Note also that the function $g(x)=-2x+1$ is decreasing and $g^{-1}(y)=(1-y)/2$ and the derivative of the inverse function is negative (which is $-1/2$). Thus the probability density function of $Y=-2X+1$ for $-1<y<1$ is

$$f_Y(y)=f_X(g^{-1}(y))\left|\frac{d}{dy}\left[g^{-1}(y)\right]\right|=2\left(-\frac{y-1}{2}\right)\left|-\frac{1}{2}\right|=\frac{1-y}{2}.$$

That is,

$$f_Y(y)=\begin{cases}\dfrac{1-y}{2} & , \quad -1<y<1\\ 0 & , \quad elsewhere.\end{cases}$$

and it is a probability density function because

$$\int_{y=-1}^{1} f_Y(y)\,dy=\frac{1}{2}\int_{y=-1}^{1}(1-y)\,dy=\frac{1}{2}\left(y-\frac{y^2}{2}\right)_{y=-1}^{1}=1.$$

<u>Example 2</u>.: Let X be a random variable with the following probability density function

$$f_X(x)=\frac{1}{\sqrt{2\pi}}\,e^{-x^2/2} \quad \text{for } x\in\mathbb{R} \tag{2}$$

and let us try to find the probability density function of $Y=X^2$. Note that the function $f_X(x)$ is an even function ($f_X(x)=f_X(-x)$). Moreover $D_X=\mathbb{R}$ and $D_Y=\mathbb{R}^+$ and therefore, $F_Y(y)=0$ for $y\le 0$. For $y>0$ the cummulative distribution function is

$$F_Y(y)=P(Y\le y)=P(X^2\le y)=P(-\sqrt{y}\le X\le\sqrt{y})=F_X(\sqrt{y})-F_X(-\sqrt{y}).$$

Thus, the probability density function of Y for $y>0$ is

$$\begin{aligned}f_Y(y)&=\frac{d}{dy}F_Y(y)=\frac{d}{dy}\left[F_X(\sqrt{y})-F_X(-\sqrt{y})\right]=\frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y})+f_X(-\sqrt{y})\right]\\&=\frac{1}{\sqrt{y}}f_X(\sqrt{y})=\frac{1}{\sqrt{y}}\frac{1}{\sqrt{2\pi}}e^{-y/2}=\frac{1}{\sqrt{\pi}\,2^{1/2}}\,y^{(1-2)/2}\,e^{-y/2}=\frac{1}{\Gamma(1/2)\,2^{1/2}}\,y^{(1-2)/2}\,e^{-y/2}.\end{aligned}$$

Therefore the probability density function pf the transformed random variable Y is

$$f_Y(y)=\begin{cases}\dfrac{1}{\Gamma(1/2)\,2^{1/2}}\,y^{(1-2)/2}\,e^{-y/2} & , \quad y>0\\ 0 & , \quad elsewhere.\end{cases} \tag{3}$$

In general the probability density function can be written for $p=1$ as

$$f_Y(y)=\begin{cases}\dfrac{1}{\Gamma(p/2)\,2^{p/2}}\,y^{(p-2)/2}\,e^{-y/2} & , \quad y>0\\ 0 & , \quad elsewhere.\end{cases} \tag{4}$$

and known as the probability density function of chi-square distribution with p degrees of freedom.

<u>A note on these probability distributions</u> (will be discussed later in details):

The random variable X with the probability density function given in (2) is known to be the standard normal random variable and denoted by $X \sim N(0,1)$ and the random variable Y with the probability density function in (3) is the chi-square random variable with 1 degrees of freedom. Similarly, if a random variable (say W) has a probability density function given in (4) we say that W is distributed as chi-square with p degrees of freedom.

In statistics, almost all statistical inferences depend on the normality assumption. If the data do not satisfy the normality assumption we use some techniques (usually trnasformations) to achieve the normality assumption. The chi-square distribution is also very important distribution in statistics which is obtained by the squares of normally distributed random variables. These distributions are also known as the sample distributions which will be discussed later.

C) <u>Mutivariate Transformations</u>:

In this part of the notes, we are going to investigate multivariate transformations. If X and Y are two random variables with joint probability (or probability density) function $f(x,y)$ we will try to find the probability (or probability density) function of $U = g_1(X,Y)$ and $V = g_2(X,Y)$. A generalization is also possible for k variate random vectors and k variate transformations. for simplicity we will only consider bivariate transformations.

Let $X_1, X_2, \ldots, X_k$ be the random variables with joint probabity (or probabilkity density) function $f(x_1, x_2, \ldots, x_k)$ and consider the following transformations

$$Y_1 = g_1(X_1,\ldots,X_k), \quad Y_2 = g_2(X_1,\ldots,X_k), \ldots, \; Y_k = g_k(X_1,\ldots,X_k).$$

Assume that the functions g_i's are invertiable and differentiable with respect to their components. We can write the Jacoien matrix as

$$J = \begin{bmatrix} \frac{\partial h_1(Y_1,\ldots,Y_k)}{\partial Y_1} & \frac{\partial h_1(Y_1,\ldots,Y_k)}{\partial Y_2} & \cdots & \frac{\partial h_1(Y_1,\ldots,Y_k)}{\partial Y_k} \\ \frac{\partial h_2(Y_1,\ldots,Y_k)}{\partial Y_1} & \frac{\partial h_2(Y_1,\ldots,Y_k)}{\partial Y_2} & \cdots & \frac{\partial h_2(Y_1,\ldots,Y_k)}{\partial Y_k} \\ \vdots & \vdots & & \vdots \\ \frac{\partial h_k(Y_1,\ldots,Y_k)}{\partial Y_1} & \frac{\partial h_k(Y_1,\ldots,Y_k)}{\partial Y_2} & \cdots & \frac{\partial h_k(Y_1,\ldots,Y_k)}{\partial Y_k} \end{bmatrix}$$

and denote $|J|$ as the absolute value of the determinant of J (that is, $|J|=|\det(J)|$) then the joint probability density function of $Y_1, Y_2, \ldots, Y_k$ is given by

$$f_{Y_1,\ldots,Y_k}(y_1,\ldots,y_k)=|J|\,f_{X_1,\ldots,X_k}(h_1(y_1,\ldots,y_k),h_2(y_1,\ldots,y_k),\ldots,h_k(y_1,\ldots,y_k)) \tag{5}$$

where $X_1 = h_1(Y_1,\ldots,Y_k)$, $X_2 = h_2(Y_1,\ldots,Y_k)$,..., $X_k = h_k(Y_1,\ldots,Y_k)$. For simplicity we will use $k=2$.

Example 1 : Let X and Y be two independent random variables with the same probability density function given below.

$$f(x)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2} \quad , x\in\mathbb{R}.$$

a) Let us consider the transformations as $U = X+Y$ and $V = X-Y$ and try to find the joint probability density function of U and V. The inverse transformations are obtained as $X=(U+V)/2$ ve $Y=(U-V)/2$ and theJacobien matrix with its determinant are

$$J=\begin{bmatrix}\dfrac{\partial X}{\partial U} & \dfrac{\partial X}{\partial V}\\ \dfrac{\partial Y}{\partial U} & \dfrac{\partial Y}{\partial V}\end{bmatrix}=\begin{bmatrix}\dfrac{1}{2} & \dfrac{1}{2}\\ \dfrac{1}{2} & -\dfrac{1}{2}\end{bmatrix} \quad \text{and} \quad \det(J)=-\frac{1}{2}.$$

Note that since the random variables X and Y are independent the joint probability density function can be written for all $x, y\in\mathbb{R}$ as

$$f(x,y)=f_X(x)\,f_Y(y)=\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}=\frac{1}{2\pi}e^{-(x^2+y^2)/2}.$$

therefore the joint probability density function of U and V for all $u,v\in\mathbb{R}$ is written as

$$\begin{aligned} f_{U,V}(u,v) &= |J|\,f_{X,Y}\left(x(u,v),y(u,v)\right)=\frac{1}{2}\frac{1}{2\pi}\exp\left(-\frac{1}{2}\left[\left(\frac{u+v}{2}\right)^2+\left(\frac{u-v}{2}\right)^2\right]\right)\\ &=\frac{1}{4\pi}\exp\left(-\frac{1}{8}\left[(u+v)^2+(u-v)^2\right]\right)\\ &=\frac{1}{4\pi}\exp\left(-\frac{1}{8}\left[2u^2+2v^2\right]\right)=\frac{1}{4\pi}\,e^{-(u^2+v^2)/4}.\end{aligned}$$

That is, the joint probability function of U and V is

$$f_{U,V}(u,v)=\frac{1}{4\pi}\,e^{-(u^2+v^2)/4} \qquad u,v\in\mathbb{R}.$$

Since joint probability density function can be written as

$$f_{U,V}(u,v) = \frac{1}{4\pi}\ e^{-(u^2+v^2)/4} \ = \frac{1}{\sqrt{4\pi}} e^{-u^2/4} \ \ \frac{1}{\sqrt{4\pi}} e^{-v^2/4} = f_U(u)\ f_V(v)$$

the random variables U ve V are independent.

b) Now let us try to find the probability density function of the transformed random variable $U = X / Y$. In order to use the equation (5) we need to define an auxiliary transformation. Let $V = Y$. First we find the joint probability density function of $U = X / Y$ and $V = Y$ and using this joint probability density function we can find the marginal probability density function of U. The inverse transformations are $X = U\,V$ ve $Y = V$ and the Jacobien matrix with its determinant are calculated as

$$J = \begin{bmatrix} \dfrac{\partial X}{\partial U} & \dfrac{\partial X}{\partial V} \\ \dfrac{\partial Y}{\partial U} & \dfrac{\partial Y}{\partial V} \end{bmatrix} = \begin{bmatrix} v & u \\ 0 & 1 \end{bmatrix} \quad \text{and } \det(J) = v .$$

therefore the joint probability density function of U and V can be written as for all $u, v \in D_{U,V}$

$$f_{U,V}(u,v) = |\,J\,|\, f_{X,Y}(x(u,v), y(u,v)) \ = \frac{|\,v\,|}{2\pi}\ \exp\left(-\frac{1}{2}[(uv)^2 + v^2)] \right)$$

$$= \frac{|\,v\,|}{2\pi}\ \exp\left(-\frac{v^2}{2}[u^2 + 1] \right).$$

Note that we want to find the probability density functiıon of U. Remember that a function $h(x)$ is even if $h(-x) = h(x)$ and if $h(x)$ is an even function we have for all $a \in \mathbb{R}^+$,

$$\int_{-a}^{a} h(x)\, dx = 2\int_{0}^{a} h(x)\, dx .$$

Therefore the joint probability density function of U and V is an even function of v. In order to find the marginal probability density function of U we integrate the joint probability density function over the range of V, D_V. The integral is obtained as (saying $a = u^2 + 1$),

$$f_U(u) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, dv = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\,v\,| e^{-a\,v^2/2} dv = \frac{2}{2\pi}\int_{0}^{\infty} v\, e^{-a\,v^2/2} dv$$

$$= \frac{1}{\pi}\int_{0}^{\infty} e^{-a\,t} dt, \qquad \text{used } t = \frac{v^2}{2}$$

$$= \frac{1}{a\,\pi}\left[-e^{-at} \Big|_{t=0}^{\infty} \right] = \frac{1}{a\,\pi} = \frac{1}{\pi}\ \frac{1}{1+u^2}$$

and therefore the probability density function of U is

$$f_U(u) = \frac{1}{\pi} \frac{1}{1+u^2} \ , \ u \in \mathbb{R} .$$

Example 2. Let X and Y be two independent random variables with the following probability density function

$$f(x) = \begin{cases} e^{-x} & , \quad x > 0 \\ 0 & , \quad d.y. \end{cases}$$

a) Let us define the transformations as $U = X + Y$ and $V = X / (X + Y)$ and try to find the joint probability density function of U and V. Note that the back transformations are $X = UV$ and $Y = U(1-V)$ and the Jacobien matrix with its determinant are calculated as

$$J = \begin{bmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{bmatrix} = \begin{bmatrix} v & u \\ (1-v) & -u \end{bmatrix} \quad \text{and} \quad \det(J) = -uv - u(1-v) = -u .$$

Since the random variables X and Y are independent the joint probability density function can be written as,

$$f(x, y) = \begin{cases} e^{-(x+y)} & , \quad x > 0, y > 0 \\ 0 & , \quad d.y. \end{cases}$$

and therefore using the equation in (5) we can write the joint probability density function of U and V for $0 < v < 1$ and $u > 0$ as

$$f_{U,V}(u,v) = | J | f_{X,Y}(x(u,v), y(u,v)) = | u | e^{-\left(uv + (u(1-v)\right)} = u e^{-u} .$$

That is the joint probability density function is

$$f_{U,V}(u,v) = \begin{cases} u e^{-u} & , \quad 0 < v < 1, u > 0 \\ 0 & , \quad d.y. \end{cases}$$

Now, it is easy to find the marginal probability density functions of U and V by using the following integrations:

$$\int_{v=0}^{1} f_{U,V}(u,v) \, dv = \int_{v=0}^{1} u e^{-u} dv = u e^{-u} \quad \text{and} \quad \int_{u=0}^{\infty} f_{U,V}(u,v) \, du = \int_{u=0}^{\infty} u e^{-u} du = 1 .$$

Thus the marginal probability density functions are

$$f_U(u)=\begin{cases} u\,e^{-u} & , \quad u>0 \\ 0 & , \quad d.y. \end{cases} \qquad f_V(v)=\begin{cases} 1 & , \quad 0<v<1 \\ 0 & , \quad d.y. \end{cases}$$

and since $f_{U,V}(u,v)=f_U(u)f_V(v)$ the random variables U are V independent.

b) let X and Y be two independent random variables with the same probability density function given below:

$$f(x)=f_Y(x)=\begin{cases} 1 & , \quad 0<x<1 \\ 0 & , \quad d.y. \end{cases}$$

Let us try to find the probability density function of $U = X\,Y$. In order to use equation (5) we need to define an auxiliary transformation. Let $V = X$ and the back transformations turn out to be $X = V$ and $Y = U / V$ and the Jacobien matrix and its determinat is calculated as

$$J=\begin{bmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{bmatrix} \quad , \quad \det(J)=-\frac{1}{v}.$$

Therefor the joint probability density function of U and V can be written as (equation in (5)) as,

$$f_{U,V}(u,v)=\begin{cases} \frac{1}{v} & , \quad 0<u<v<1 \\ 0 & , \quad d.y. \end{cases}$$

and the marginal probability density function of U is calculated from the integral as

$$\int_{v\in D_V} f_{U,V}(u,v)\,dv=\int_{v=u}^{1}\frac{dv}{v}=-\ln(u).$$

Therefore the probability density function of U is

$$f_U(u)=\begin{cases} -\ln(u) & , \quad 0<u<1 \\ 0 & , \quad d.y. \end{cases}$$

c) Let X_1, X_2, X_3 be three random variables with the joint probability density function

$$f_{X_1,X_2,X_3}(x_1,x_2,x_3)=\begin{cases} 6\,e^{-x_1-x_2-x_3} & , \quad 0<x_1<x_2<x_3<\infty \\ 0 & , \quad d.y. \end{cases}$$

Suppose we want to find the joint probability density function of $U_1 = X_1$, $U_2 = X_2 - X_1$ and $U_3 = X_3 - X_2$. Note that the back transformations are found to be

$$X_1 = U_1 \quad X_2 = U_1 + U_2, \; X_3 = U_1 + U_2 + U_2$$

and the Jacobien matrix ant its determinant are calculated as

$$J=\begin{bmatrix}\frac{\partial X_1}{\partial U_1} & \frac{\partial X_1}{\partial U_2} & \frac{\partial X_1}{\partial U_3}\\ \frac{\partial X_2}{\partial U_1} & \frac{\partial X_2}{\partial U_2} & \frac{\partial X_2}{\partial U_3}\\ \frac{\partial X_3}{\partial U_1} & \frac{\partial X_3}{\partial U_2} & \frac{\partial X_3}{\partial U_3}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 1 & 1 & 0\\ 1 & 1 & 1\end{bmatrix} \quad , \qquad \det(J)=1 .$$

therefore the joint probability density function of U_1, U_2, U_3 is

$$f_{U_1,U_2,U_3}(u_1,u_2,u_3)=\begin{cases}6\, e^{-3u_1-2u_2-u_3} & , \quad u_i>0, i=1,2,3\\ 0 & , \quad d.y.\end{cases}$$

Note that eventhogh the random variables X_1, X_2, X_3 are not independent since

$$f_{U_1,U_2,U_3}(u_1,u_2,u_3)=f_{U_1}(u_1)f_{U_2}(u_2)f_{U_3}(u_3)$$

the random bvariables U_1, U_2, U_3 are independent.

d) Let $X_1, X_2,..., X_n$ be independent random variables with the same probability density function

$$f(x)=\begin{cases}\frac{1}{\theta} & , \quad 0<x<\theta\\ 0 & , \quad d.y.\end{cases}$$

Now we want to find the probability density function of $U=\max\{X_1, X_2,..., X_n\}$. Note that we can not use the formula in the equation in (5). Therefore, we need to calculate its cumulative distribution function. Note that the range of U is the same as the range of X's. Therefore, $F_U(u)=0$ for $u\le 0$ and $F_U(u)=1$ for $u\ge\theta$. For $0<u<\theta$

$$F_U(u)=P(U\le u)=P(\max\{X_1,X_2,...,X_n\}\le u)=P(X_1\le u, X_2\le u,..., X_n\le u)$$

$$=\prod_{i=1}^{n}P(X_i\le u)=\left(P(X_1\le u)\right)^n=\left(\int_{x=0}^{u}\frac{1}{\theta}dx\right)^n=\frac{1}{\theta^n}u^n .$$

Thus, the cumulative distribution function and probability density function (which is the derivative of the cumulative distribution) of U are

$$F_U(u)=\begin{cases}0 & , \quad u\le 0\\ \frac{u^n}{\theta^n} & , \quad 0<u<\theta\\ 1 & , \quad u\ge\theta\end{cases} \qquad \text{and} \qquad f_U(u)=\frac{dF_U(u)}{du}=\begin{cases}\frac{n}{\theta^n}u^{n-1} & , \quad 0<u<\theta\\ 0 & , \quad elsewhere.\end{cases}$$

e) Let X ve Y be two independent random variables with the same probability density function given below:

$$f_X(x)=\begin{cases}e^{-x} & , \quad x>0\\ 0 & , \quad d.y.\end{cases}$$

Suppose we want to calculate the probability density functions of $U = \max(X,Y)$ and $V = \min(X,Y)$. Here, we are going to calculate the distribution functions of both random variables.

First let us find the cumulative distribution function of U. Note that $F_U(u) = 0$ for $u < 0$ and for $u \geq 0$,

$$F_U(u) = P(U \leq u) = P(\max(X,Y) \leq u) = P(X \leq u, Y \leq u)$$
$$= P(X \leq u)P(Y \leq u) = [P(X \leq u)]^2 = (1-e^{-u})^2.$$

thus, the cumualtive distribution function and the probability density function (derivative of $F_U(u)$) of U are

$$F_U(u) = \begin{cases} 0 & , \quad u < 0 \\ (1-e^{-u})^2 & , \quad u \geq 0 \end{cases} \qquad \text{and} \qquad f_U(u) = \begin{cases} 2e^{-u}(1-e^{-u}) & , \quad u > 0 \\ 0 & , \quad d.y. \end{cases}$$

In a similar way, we can calculate the cumulative distribution function of V. Note that $F_V(v) = 0$ for $v < 0$ and for $v \geq 0$

$$F_V(v) = P(V \leq v) = P(\min(X,Y) \leq v) = 1 - P(\min(X,Y) > v)$$
$$= 1 - P(X > v, Y > v) = 1 - P(X > v)P(Y > v) = 1 - [P(X > v)]^2 = 1 - e^{-2v}$$

and thus the cumulative distribution function and probability density function of V are given below: nin dağılım fonksiyonu da

$$F_V(v) = \begin{cases} 0 & , \quad v < 0 \\ 1-e^{-2v} & , \quad v \geq 0 \end{cases} \qquad \text{and} \qquad f_V(v) = \begin{cases} 2e^{-2v} & , \quad v > 0 \\ 0 & , \quad d.y. \end{cases}$$

Discrete case: For discrete case, the probability function of a transformed random variable can be found directly by calculating the related probabilities. There is also an easier way (generating function technique) the we are going to study next. here is an example how to find the probability distribution of a transformed random variables for discrete case.

Example: Let X and Y be two independent random variables with the following probability distribution function:

$$P(X = x) = P(Y = x) = e^{-\lambda}\lambda^x / x! \quad , x = 0,1,2,\ldots \text{ and } \lambda > 0.$$

Suppose we want to find the probability distribution of $U = X + Y$. Obviously the range of U is the same as the range of X (or Y). Therefore, the probability distribution of U can be calculated as for $u = 0,1,2,3,\ldots$

$$P(U=u)=P(X+Y=u)=\sum_{y=0}^{u}P(X+Y=u\mid Y=y)P(Y=y)=\sum_{y=0}^{u}P(X+y=u)P(Y=y)$$

$$=\sum_{y=0}^{u}P(X=u-y)P(Y=y)=\sum_{y=0}^{u}\left(\frac{e^{-\lambda}\lambda^{u-y}}{(u-y)!}\right)\left(\frac{e^{-\lambda}\lambda^{y}}{y!}\right)=\sum_{y=0}^{u}\frac{u!}{u!}\left(\frac{e^{-\lambda}\lambda^{u-y}}{(u-y)!}\right)\left(\frac{e^{-\lambda}\lambda^{y}}{y!}\right)$$

$$=\frac{e^{-2\lambda}}{u!}\sum_{y=0}^{u}\left(\frac{u!}{y!(u-y)!}\right)\lambda^{y}\lambda^{u-y}=\frac{e^{-2\lambda}}{u!}\sum_{y=0}^{u}\binom{u}{y}\lambda^{y}\lambda^{u-y}=\frac{e^{-2\lambda}}{u!}(\lambda+\lambda)^{u}=\frac{e^{-2\lambda}(2\lambda)^{u}}{u!}.$$

Note that the probability function of U is similar to the probability function of X (or Y). The only difference we have 2λ instead of λ and therefore the probability distribution of U is

$$P(U=u)=e^{-2\lambda}(2\lambda)^{u}/u! \quad , \ u=0,1,2,\ldots \ .$$

Generating Function Technique: We have studied some of generating functions in the previous sections. If X is a random variable with probability (or probability density) function $f(x)$, the moment generating function of X can be calculated as $M_X(t)=E(e^{tX})$. Moreover, X and Y are two independent random variables with moment generating functions $M_X(t)$ and $M_Y(t)$ respectively, the moment generating function of $U=aX+bY$ is

$$M_U(t)=M_{aX+bY}(t)=E(e^{t(aX+bY)})=E(e^{taX})E(e^{tbY})=M_X(at)M_Y(bt).$$

If the moment generating function is similar to a moment generating function of a special random variable then their distributions are similar.

Example: **a)** Let X and Y be two independent random variables with the same probability function given below:

$$P(X=x)=e^{-\lambda}\lambda^{x}/x! \ , \ x=0,1,2,\ldots, \ \lambda>0.$$

The moment genarating function of X (or Y) is

$$M_X(t)=E(e^{tX})=\sum_{x=0}^{\infty}e^{tx}P(X=x)=\sum_{x=0}^{\infty}e^{tx}e^{-\lambda}\lambda^{x}/x!=e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda e^{t})^{x}}{x'}=e^{-\lambda}e^{\lambda e^{t}}=e^{\lambda(e^{t}-1)}.$$

Suppose we want to find the distribution of $U=X+Y$. Since X and Y two independent random variables with the same moment generating function $M_X(t)=e^{\lambda(e^{t}-1)}$ the moment genarating function of U can be written as

$$M_U(t)=M_{X+Y}(t)=M_X(t)M_Y(t)=(e^{\lambda(e^{t}-1)})(e^{\lambda(e^{t}-1)})=e^{2\lambda(e^{t}-1)}$$

which is the same moment generating function of X (or Y) except we have 2λ instead of λ. Therefore their probability distributions (U and X) are similar. All we need to do is to put 2λ for λ. That is, the probability distribution of U is

$$P(U=u)=e^{-2\lambda}(2\lambda)^u/u! \quad ,u=0,1,2,\ldots.$$

Note that this is the same probability function as we have calculated directly.

b) Let X and Y be two independent random variables with the same probability function given below:

$$P(X=x)=P(Y=x)=p^x q^{1-x}, x=0,1\,;\, 0<p<1 \text{ and } q=1-p$$

Since, their probability functions are the same their moment generating functions are also the same. the moment generating function of X is calculated as

$$M_X(t)=E(e^{tX})=\sum_{x=0}^{1} e^{tx}P(X=x)=q+pe^t.$$

Now, we want to find the distribution of $U=X+Y$. Since they are independent random variables, the moment generating function of U is calculated as

$$M_U(t)=M_{X+Y}(t)=M_X(t)M_Y(t)=(q+pe^t)(q+pe^t)=(q+pe^t)^2.$$

Now, consider a random variable Z with the probability function

$$P(Z=x)=\binom{2}{x}p^x q^{2-x}\ ,\ x=0,1,2\ ,\ 0<p<1 \text{ and } q=1-p$$

and the moment generating function of Z is

$$M_Z(t)=E(e^{tZ})=\sum_{x=0}^{2} e^{tx}\binom{2}{x}p^x q^{2-x}=\sum_{x=0}^{2}\binom{2}{x}(pe^t)^x q^{2-x}=(q+pe^t)^2$$

which is the same function as $M_U(t)$ and therefore their distributions are similar. That is, the probability function of U is

$$P(U=u)=\binom{2}{u}p^u q^{2-u}\ ,\ u=0,1,2.$$

c) let X and Y be two independent random variables with the following probability density functions. The probability distributions are given as for $\mu_x,\mu_y\in\mathbb{R}$ and $\sigma_x>0$, $\sigma_y>0$,

$$f_X(x)=\frac{1}{\sqrt{2\pi\sigma_x^2}}\exp\left(-\frac{1}{2\sigma_x^2}(x-\mu_x)^2\right)\ ,\ x\in\mathbb{R}$$

$$f_Y(y)=\frac{1}{\sqrt{2\pi\sigma_y^2}}\exp\left(-\frac{1}{2\sigma_y^2}(y-\mu_y)^2\right)\ ,\ y\in\mathbb{R}.$$

Their moment generating functions are calculated as

$$M_X(t)=\exp\left(t\mu_x+\frac{t^2\sigma_x^2}{2}\right) \quad \text{and} \quad M_Y(t)=\exp\left(t\mu_y+\frac{t^2\sigma_y^2}{2}\right).$$

The moment generating function of $U=X+Y$ is calculated as

$$M_U(t) = M_{X+Y}(t) = M_X(t)M_Y(t) = \exp\left(t\,\mu_x + \frac{t^2\sigma_x^2}{2}\right)\exp\left(t\,\mu_x + \frac{t^2\sigma_x^2}{2}\right)$$

$$= \exp\left(t(\mu_x + \mu_y) + \frac{t^2(\sigma_x^2 + \sigma_x^2)}{2}\right) = \exp\left(t\,\mu + \frac{t^2\sigma^2}{2}\right)$$

where $\mu = \mu_x + \mu_y$ and $\sigma^2 = \sigma_x^2 + \sigma_y^2$. As it is seen the moment generatin function of U is similar to the moment generating function of X (or Y) and therefore their probability density functions are also similar. That is, the probability density function of $U = X + Y$ is

$$f_U(u) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(u-\mu)^2\right) \quad , \quad u \in \mathbb{R}$$

where $\mu = \mu_x + \mu_y$ and $\sigma^2 = \sigma_x^2 + \sigma_y^2$.